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On Information-Processing Abilities of Chaotic Dynamical Systems

Nolan W. Evans,[1] **Reinhard Illner,**[2] **and Hon C. Kwan**[3]

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A mechanism is suggested to explain the information processing abilities of simple natural brains, which, by experimental evidence, display behavior like chaotic dynamical systems while at rest. The Lorenz system of equations is dealt with as a case study, and a comparison of the suggested mechanism with the standard theory of neural networks is made.

KEY WORDS: Information processing; chaotic dynamical systems; Lorenz attractor.

1. INTRODUCTION

We are concerned in this paper with neural networks considered from a dynamical systems point of view. Many neural networks can be modeled as dynamical systems, and their computational process is the evolution of an input—the initial state—to a stable fixed point or limit cycle—the output or result of the computation (see, e.g., refs. 13, 14, 20, and 21). Networks currently used and studied are autonomous, dissipative, and nonchaotic systems—phase space is contracted under the evolution.

Such systems are, for experimental and theoretical reasons, inadequate to give a satisfactory model of real, biological brains. From a theoretical point of view, it seems unreasonable that just the setting of a particular state (the input) should lead to an asymptotic state (the output) in which the system would then stay indefinitely. Such a system would be unable to forget and would in some sense be unable to prepare itself for new input. It would need to be reset from the outside for the next computation and could not be called "alive" in any sense of the word. On the experimental

[1] Department of Mathematics, University of Guelph, Guelph, Ontario, Canada.
[2] Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.
[3] Department of Physiology, University of Toronto, Toronto, Ontario, Canada.

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side, even the brains of relatively primitive animals (like invertebrates; see, e.g., Miptsos *et al.*[18]) display dramatically different behavior: When at rest, i.e., when no inputs are given, such brains show frantic activity. Measurements of power spectra and correlation dimensions of neuronal activities suggest that a good model of such a system must allow for chaotic dynamical behavior, especially "at rest."[17,22] Inputs typically transform the system into periodic oscillations.[4,10,11]

There are efforts to modify traditional logical neural networks to make them compatible with some of the additional features observed in real biological brains, e.g., periodic oscillations and low firing rates.[1,5] The networks introduced in these references are possibly examples for the general scenario which we discuss below.

There is growing consensus that chaos is actually an essential ingredient in the dynamics of biological neural nets.[3,19,22,23,25] Actually, chaos may just be a generic phenomenon related to the high dimension of phase space; the crucial question is how the system will respond to inputs. If the rest state of the system is on, say, a strange attractor, then it does not make sense to consider initial values as inputs. Rather, an input will be an outer force which acts temporarily on the system. While this force is active it causes a structural change in the phase diagram of the system, and so the system may well settle in a fixed point or on a limit cycle (whose location will specifically depend on the direction and size of the input). Upon removal of the outer force, the network relaxes and returns to its dynamic rest state, the strange attractor.

We suggest that a resting biological neural network does not display many stable steady states (like, e.g., the Hopfield net for symmetric connections[13]), but will typically settle on a complex chaotic attractor. Even though one may have sensitive dependence on initial conditions, the system is unable to distinguish initial values, because they are all pulled to the same set. However, structural changes in the phase space may well be specific to the applied force and thus lead to a fixed point or limit cycle. Therefore, outer forces, not initial data, must serve as inputs.

Suppose the model consists of N neurons, and let $\mathbf{x} = (x_1, ..., x_N)$ be the vector of postsynaptic potentials (neural states as indicated by action potential frequencies); then the undisturbed system would be modeled by a system of differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1}$$

Assume next that a sensory input, represented by a unit vector $\mathbf{y} \in S^{N-1}$, with strength $\alpha \geqslant 0$, is active during a time interval $[t_1, t_2]$. Equation (1) changes into

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \alpha \cdot \mathbf{y} \cdot \chi_{[t_1, t_2]}$$

where $\chi_{[t_1, t_2]}$ denotes the characteristic function of the time interval $[t_1, t_2]$. For small α, the vector field on the right will certainly change while the input is active, but the attractor will presumably be stable enough to persist structurally—the neural system simply does not recognize small inputs. For large α, there are two possibilities: The first is that there may be directions $\mathbf{y}$ which are still not "recognized"—the chaotic attractor is distorted, but not destroyed. One could say that the system is not designed to "see" $\mathbf{y}$. The second is that the attractor may undergo a structural change such as a deformation into a limit cycle or a stable fixed point. The location of the fixed point (or limit cycle) will depend on $\mathbf{y}$ and α, and so the system gives a clear "response" to $\alpha\mathbf{y}$.

2. A CASE STUDY: THE LORENZ SYSTEM

The Lorenz equations are a famous system of nonlinear ordinary differential equations, suggested by Lorenz in 1963[16]:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma(x-y) \\ \frac{dy}{dt} &= -y + rx - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned} \tag{2}$$

This system has been extensively studied, especially the structure of its attracting sets. For details, we refer the reader to the book by Sparrow[24] or to the article by Lanford.[15] We begin our study by listing some of the crucial properties of the system.

(a) Every trajectory enters a compact invariant set in finite time. This follows because if we define

$$u(t) := x(t)^2 + y(t)^2 + [z(t) - r - \sigma]^2$$

then there exist positive constants c_1 and c_2 such that

$$\dot{u} \leqslant -c_1 u + c_2 \tag{3}$$

The structure of the system of equations is essential for this property: Note that there are no terms of third order (like xyz, or x^2y) if we multiply the three equations by x, y, and z, respectively, and add up.

(b) The system contracts phase space, because

$$\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}+\frac{\partial \dot{z}}{\partial z}=\operatorname{div} \mathbf{F}(x, y, z)=-(\sigma+b+1)$$

and σ and b are nonnegative.

The behavior of solutions to (1) is strongly dependent on the value of the parameters b, σ, and r. Typical values for which the famous Lorenz attractor exists are $b=8/3$, $\sigma=10$, and $r=28$. Properties (a) and (b) imply that every solution trajectory tends toward an attracting set A. This set is of Lebesgue measure zero in $\mathbb{R}^3$ (actually, it is of reduced Hausdorff dimension), and its structure depends on the parameters b, σ, and r. For the above values, the attractor is the famous Lorenz "butterfly."

Even though they are unstable, the fixed points of the system (1) play a pivotal role for the asymptotic behavior of solutions. These fixed points are $Q=(0,0,0)$, $P_1=([b(r-1)]^{1/2}, [b(r-1)]^{1/2}, r-1)$, and $P_2=(-[b(r-1)]^{1/2}, -[b(r-1)]^{1/2}, r-1)$. For the given values of the parameters σ, b, and r, all these fixed points are unstable. Q has a two-dimensional stable and one-dimensional unstable manifold, whereas P_1 and P_2 have two-dimensional unstable and one-dimensional stable manifolds. Roughly speaking, the Lorentz attractor exists as a connection from the two-dimensional unstable manifold of P_1 to the one-dimensional stable manifold of P_2, and vice versa (for more details see refs. 12 and 24). The result is that a trajectory will oscillate back and forth between the unstable manifolds attached to P_1 and P_2. (A typical trajectory moves outward in a spiral near the unstable manifold of P_1 until it is attracted near the stable manifold of P_2 toward P_2. It then spirals outward from P_2, until the stable manifold of P_1 begins to attract it, and so on. In a neighborhood of P_1 and P_2 this behavior can be predicted via linear stability analysis,[15] but far away from the fixed points the system is much harder to predict. The attractor is called strange because all evidence suggests that it is not a limit cycle or quasiperiodic attractor; it displays sensitive dependence on the initial conditions, at least on the computational and heuristic level. However, to our knowledge, it has never been rigorously shown that the Lorenz attractor is not just a limit cycle with a very long period.

Be that as it may, the issue is of little relevance for our present objective. The important feature is that P_1 and P_2 are actually attracting in some sense, but due to their instability, the system is unable to "decide" between them. This inability is chaotic inasmuch as the amount of time a trajectory spends near either point appears to be random, even after the trajectory has relaxed toward the attractor for a very long time. Experiments suggest that in the long run, a trajectory will spend about half of its time on either side of the plane $x+y=0$.

We suggest that the Lorenz system can be interpreted as a neural network "at rest." More specifically, the chaotic motion along the Lorenz attractor is the system state *without* external inputs; the system is ready to process information in the sense that it can make a decision between either side of the plane $x + y = 0$.

Suppose that an external input is active during the interval $[t_1, t_2]$. For simplicity, we model an input as a constant impulse in the direction $\mathbf{u} = (A, B, C)$, $\|\mathbf{u}\| = 1$, with strength $\alpha > 0$, i.e., the system under consideration is

$$\begin{aligned} \dot{x} &= \sigma(y - x) + \alpha A \chi_{[t_1, t_2]} \\ \dot{y} &= rx - y - xz + \alpha B \chi_{[t_1, t_2]} \\ \dot{z} &= -bz + xy + \alpha C \chi_{[t_1, t_2]} \end{aligned} \tag{4}$$

As an example, let $A = 0$, $C = 0$, and $B = 1$ or -1. Fixed points of the perturbed system must then satisfy the equations $x = y$, $z = x^2/b$, $(r - 1)\,x - x^3/b = \pm\alpha$. For $\alpha = 0$, we obtain the hyperbolic fixed points of the ordinary Lorenz system. Their x values are given as intersections of the polynomial $p(x) = (r - 1)\,x - x^3/b$ with the x axis and lead to the x values of P_1 and P_2.

For the genuinely perturbed system ($\alpha > 0$), we first focus on the case $A = 0$, $B = 1$, $C = 0$. To get the x values of Q, P_1, and P_2, we have to solve

$$x(r - 1) - x^3/b + \alpha = 0$$

If we set $c = r - 1$ and $\alpha_0 = 2/3c(bc/3)^{1/2}$, then the equation has three real solutions for $0 \leqslant \alpha < \alpha_0$, two real solutions for $\alpha = \alpha_0$, and one real and two complex conjugate solutions for $\alpha > \alpha_0$ (see Fig. 1). For $r = 28$ and $b = 8/3$, we find $\alpha_0 = 36 \cdot \sqrt{6} \approx 88.18$.

Let x_0, x_1, and x_2 be the x coordinates of Q, P_1, and P_2, respectively (these fixed points depend now of course on α, but we suppress this dependence in the notation). As is clear from Fig. 1, Q and P_2 approach each other as α increases, merge at $\alpha = \alpha_0$, and disappear for $\alpha > \alpha_0$. It can

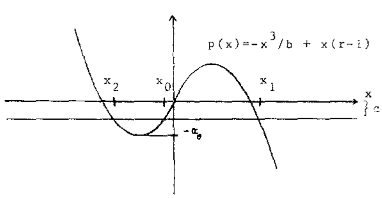

Fig. 1

be shown analytically that $Q(\alpha_0) = P_2(\alpha_0)$ is unstable, but $P_1(\alpha_0)$ is stable (we present this calculation, which was done by Pauline van den Driessche, in the Appendix). In fact, our numerical experiments suggest that $P_1(\alpha)$ becomes stable for values of α much smaller than α_0 (and Q and P_2 are unstable for $0 \leqslant \alpha < \alpha_0$).

If $A = 0$, $B = -1$, and $C = 0$, $P_2(\alpha)$ becomes the stable fixed point, while $P_1(\alpha)$ and $Q(\alpha)$ are unstable and merge at $\alpha = \alpha_0$ (they vanish for $\alpha > \alpha_0$).

Of course, this linear stability analysis of fixed points does not imply that solution trajectories will always converge to the stable fixed point (the stability analysis is valid only in sufficiently small neighborhoods of fixed points, because we are dealing with a nonlinear system). However, our numerical evidence suggests that the stable fixed point becomes, for α within reasonable bounds, a global attractor for the system. This appears to be true as well when $A \neq 0$ and $C \neq 0$, as long as $|B|$ is large enough: The sign of B determines the "choice" of the system between P_1 and P_2.

Remarks. Of course, the stable fixed point depends on the input; however, for a wide range of inputs, P_1 remains in the range $x + y > 0$, while P_2 remains in $x + y < 0$. Therefore, the system can answer the question: "Is the y coordinate of the input positive or negative?." This is a very limited information processing ability, but not too much should be expected from a system with only three neurons.

Also, as soon as the input is removed, the system returns to the Lorenz attractor. In order to keep the result in memory, the fixed point P_1 or P_2, whichever was the result of the calculation, would have to be fed to a more conventional neural network (a perceptron, for example) which would use the outputs of the perturbed Lorenz system as inputs.

3. A NEURAL NET MODELED BY THE LORENZ SYSTEM

It is relatively easy to design a 3-neuron network whose frequencies of action potentials satisfy the Lorenz system of equations. Such a net is shown in Fig. 2. The synaptic connections marked III and IV are examples of axoaxonal synapses. This situation is actually observed in real biological networks.[8,9] The enlarged picture is roughly as shown in Fig. 3.

It is experimentally observed[2,7] that the synaptic effect of the synaptic connection 1 on the target neuron is in this situation dependent on the modulatory effect of connection 2. This can be used to implement the quadratic terms on the rhs of (2), and it is a mechanism which implies nonlinear feedback inside the system. The other terms in (2) are of more conventional origin: Leakage ($-\sigma x$, $-y$, and $-bz$) and connections with constant

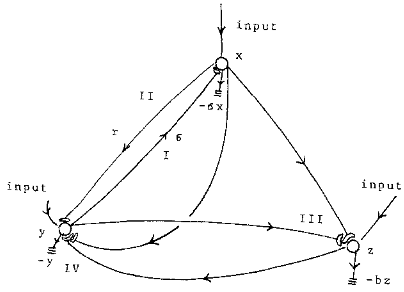

Fig. 2

connection strength (I, with connection strength σ, and II, with strength r, both excitatory).

In the absence of external inputs, the system always remains inside a bounded domain in phase space, as follows from (3). The specific design which enables the system to perform this "balancing act" is the fact that the quadratic terms compensate each other in some sense [such that (3) holds]; they are divided in just the right way into presynaptic facilitation and presynaptic inhibition. No sigmoid nonlinearity is necessary to prevent "blowup."

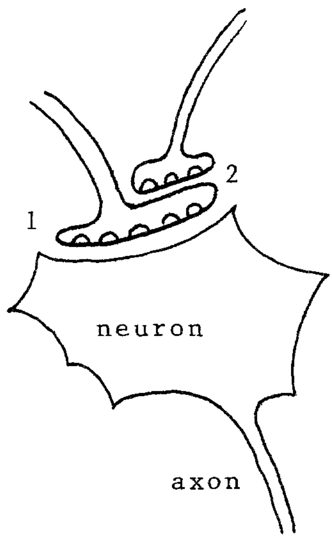

Fig. 3

We suggest that graphs like the one in Fig. 3 and the observation of the compensation property just mentioned should be used to design more complicated neural networks modeled by systems of nonlinear ordinary differential equations. We plan to address this idea in future work.

4. NUMERICAL EXPERIMENTS

Figures 4–11 show plots of approximate solutions to the Lorenz system. The solution trajectories are projected into the plane $x = y$. The line $z = 0$ is the horizontal line on top, and the z axis points down in the middle of each figure. In every experiment, an initial value (x_0, y_0, z_0) was chosen at random. The unperturbed system was solved with an arbitrary but fixed number of iteration steps N. At the end of this procedure, the perturbation was added [vector (A, B, C), multiplied by strength α] and another M iteration steps were done. We list x_0, y_0, and z_0 with most figures, as well as N, M, A, B, C, and α. The numerical method used was a simple improved Euler scheme with step size 0.004.

APPENDIX. LINEARIZED STABILITY ANALYSIS FOR THE PERTURBED FIXED POINTS

First, we briefly present the well-known stability analysis of the fixed points of the unperturbed Lorenz system. The Jacobian matrix of the rhs of (2) is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

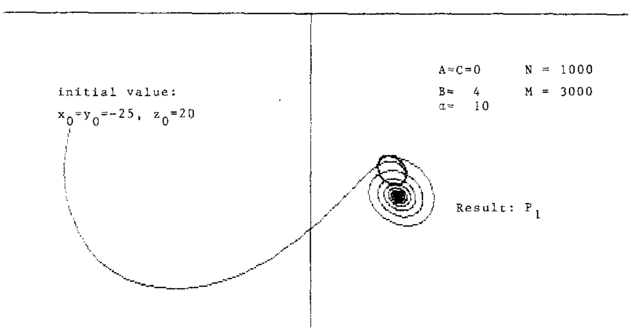

Fig. 4. Initial values: $x_0 = y_0 = -25$, $z_0 = 20$. Here $A = C = 0$, $B = 4$, $\alpha = 10$, $N = 1000$, and $M = 3000$. Result: P_1.

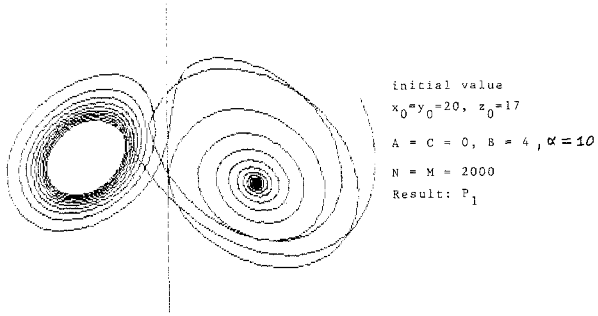

Fig. 5. Initial values: $x_0 = y_0 = 20$, $z_0 = 17$. $A = C = 0$, $B = 4$, $\alpha = 10$, $N = M = 2000$. Result: P_1.

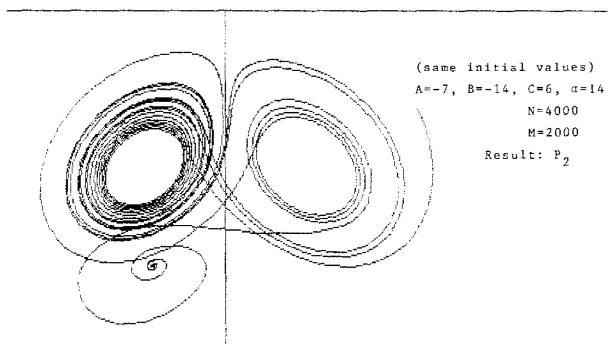

Fig. 6. Same initial values. $A = -7$, $B = -14$, $C = 6$, $\alpha = 14$, $N = 4000$, $M = 2000$. Result: P_2.

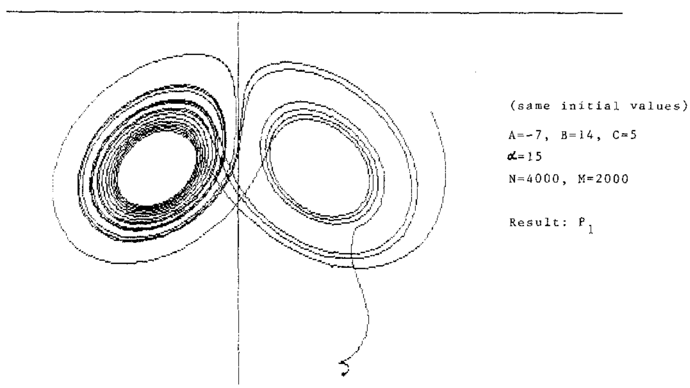

Fig. 7. Same initial values. $A = -7$, $B = 14$, $C = 5$, $\alpha = 15$, $N = 4000$, $M = 2000$. Result: P_1.

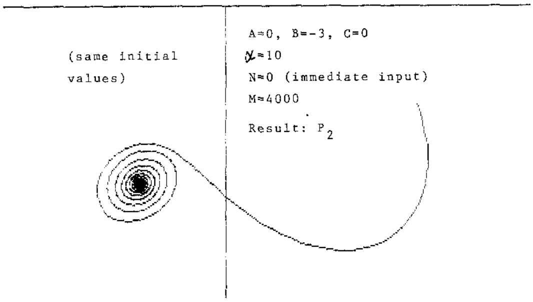

Fig. 8. Same initial values. $A=0$, $B=-3$, $C=0$, $\alpha=10$, $N=0$ (immediate input), $M=4000$. Result: P_2.

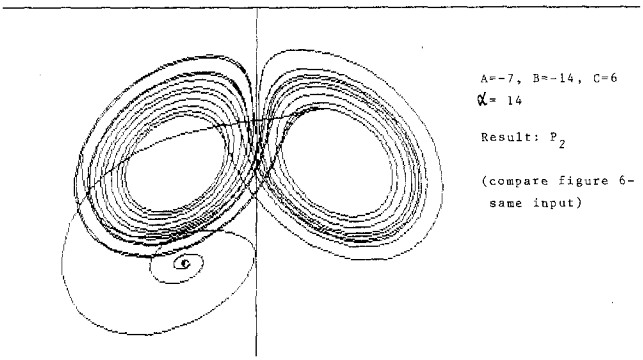

Fig. 9. Here $A=-7$, $B=-14$, $C=6$, $\alpha=14$. Result: P_2. Compare Fig. 6—same input.

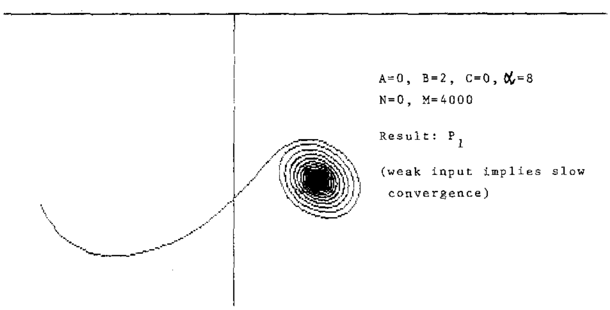

Fig. 10. Here $A=0$, $B=2$, $C=0$, $\alpha=8$, $N=0$, $M=4000$. Result: P_1. Weak input implies slow convergence.

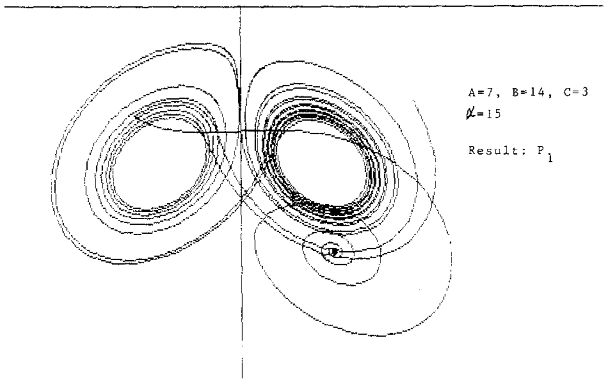

Fig. 11. Here $A=7$, $B=14$, $C=3$, $\alpha=15$. Result: P_1.

For any of the fixed points Q, P_1, or P_2 we have $y=x$, and $z=x^2/b$, so

$$J=\begin{pmatrix} -\sigma & \sigma & 0 \\ r-x^2/b & -1 & -x \\ x & x & -b \end{pmatrix}$$

For $x=0$ and $r>1$, one of the three (real) roots of the characteristic polynomial of this matrix is positive, so we have a one-dimensional unstable manifold for Q. For $x=+(-)[b(r-1)]^{1/2}$ (corresponding to P_1 or P_2) and $r=28$, $\sigma=10$, $b=8/3$, the characteristic polynomial has a complex conjugate pair of roots with positive real parts, so we have a two-dimensional unstable manifold.

For the perturbed system, we focus on the case $A=0$, $B=1$, $C=0$, and $\alpha=\alpha_0$. Then there are two fixed points (Q and P_2 are identical). With $c=r-1$, we find the x values of these fixed points by solving the equation

$$x^3-3\left(\frac{bc}{3}\right)x-2\left(\frac{bc}{3}\right)^{3/2}=0$$

The solutions are $x_1=2(bc/3)^{1/2}$ and $x_2=-(bc/3)^{1/2}$, and we claim that P_1 is stable, whereas P_2 is unstable. The Jacobian matrix for the perturbed system is the same as for the unperturbed system. Inserting x_2 into the matrix, we can calculate explicitly the roots of the characteristic polynomial

$$\lambda^3+(b+\sigma+1)\,\lambda^2+\left[\frac{c}{3}(b-2\sigma)+b(\sigma+1)\right]\lambda$$

All of them are real, and one is positive. For x_1, the characteristic polynomial is

$$\lambda^3 + (b+\sigma+1)\,\lambda^2 + \left[b(\sigma+1) + \frac{c}{3}(\sigma+4b)\right]\lambda + 3\sigma bc$$

and the explicit calculation of the roots is harder. However, we can use the Routh–Hurwitz criterion[6] to show that all roots must have a negative real part: Indeed, for $c=27$, $\sigma=10$, and $b=8/3$ we readily check that

$$(b+\sigma+1)\left[b(\sigma+1) + \frac{c}{3}(\sigma+4b)\right] > 3\sigma bc$$

and by Routh–Hurwitz, this implies stability of the fixed point.

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